1. (a). If $y = \sin\left(m\sin^{-1} x\right)$**, prove that** $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$ **3**

Solution: Given, $y = \sin\left(m\sin^{-1} x\right)$ … (1)

Differentiating (1) both sides *w.r.t.*, *x*, we get

$$y_1 = \cos\left(m\sin^{-1} x\right)\left[\frac{m}{\sqrt{1-x^2}}\right]$$

$$\sqrt{1-x^2}\; y_1 = m\cos\left(m\sin^{-1} x\right)$$

Squaring both sides, we get

$$\left(1-x^2\right) {y_1}^2 = m^2 \cos^2\left(m\sin^{-1} x\right)$$

$$\Rightarrow \quad \left(1-x^2\right) {y_1}^2 = m^2 \left[1-\sin^2\left(m\sin^{-1} x\right)\right]$$

$$\Rightarrow \quad \left(1-x^2\right) {y_1}^2 = m^2 \left(1-y^2\right) \qquad \text{[From (1)]}$$

Again differentiating (2) both sides, *w.r.t.*, *x*, we get

$$\left(1-x^2\right)2y_1.y_2 - 2x\,{y_1}^2 = m^2\left(-2y\,y_1\right)$$

$$\Rightarrow \quad \left(1-x^2\right) y_2 - x\,y_1 + m^2\, y = 0$$

$$\Rightarrow \quad \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0$$

Proved

(b).The equation of the tangent at the point (2, 3) of the curve $y^2 = ax^3 + b$ is $y = 4x – 5$. Find the value of *a* and *b*. **4**

Solution: Given: $y^2 = ax^3 + b$ …(1)

Differentiating *w.r.t.*, *x* we get

$$2y\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3a(2)^2}{2(3)} = 2a \qquad \text{at } (2, 3)$$

The equation of tangent at (2, 3) is

$$y-3=\frac{dy}{dx}(x-2)$$

$\Rightarrow$ $$y-3=2a(x-2)$$

$\Rightarrow$ $$y=2ax-4a+3 \qquad \ldots (2)$$

But given the equation of tangent is $y=4x-5$ … (3)

Equation (2) and (3) represent the same line, and then we have

$$\frac{1}{1}=\frac{2a}{4}=\frac{-4a+3}{-5}$$

$\Rightarrow$ $$\frac{2a}{4}=1 \Rightarrow a=2 \quad \text{and} \quad \frac{-4a+3}{-5}=1 \Rightarrow a=2$$

At the point (2, 3), then from equation (1), we get

$$3^2=2(2)^3+b \quad \Rightarrow b=9-16=-7$$

Thus, $\boxed{a=2, b=-7}$ **Answer**

c). Evaluate $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$ **5**

Solution: Let $$I=\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$$

$$=\int_0^{\pi/2} \frac{\sin 2x}{\left(\sin^2 x\right)^2+\left(1-\sin^2 x\right)^2}\,dx$$

Putting $\sin^2 x=t \Rightarrow 2\sin x\cos x\,dx=dt$ *i.e.* $\sin 2x\,dx=dt$

Upper Limit: $t=\sin^2\left(\frac{\pi}{2}\right)=1$

Lower Limit: $t=\sin^2(0)=0$

$\therefore$ $$I=\int_0^1 \frac{dt}{t^2+(1-t)^2}$$

$$=\int_0^1 \frac{dt}{t^2+1+t^2-2t}=\int_0^1 \frac{dt}{2t^2-2t+1}$$

$$=\frac{1}{2}\int_0^1 \frac{dt}{t^2-t+\frac{1}{2}}=\frac{1}{2}\int_0^1 \frac{dt}{\left(t^2-t+\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)}$$ [Add and subtract 1/4 in the denominator]

$$=\frac{1}{2}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$=\frac{1}{2}\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left[\frac{t-\frac{1}{2}}{\frac{1}{2}}\right]_0^1=\left[\tan^{-1}(2t-1)\right]_0^1$$

$$=\tan^{-1}(1)-0=\frac{\pi}{4}$$

Thus $\boxed{\int_0^{\pi/2}\frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx=\frac{\pi}{4}}$ **Answer**

2. a) Expand by Maclaurin's theorem $e^{x\cos x}$ as far as term x^3. **3**

Solution: Suppose $y=e^{x\cos x}$ … (1)

Differentiating w.r.t. x, successively, we get

$\Rightarrow \quad y_1=e^{x\cos x}\left[-x\sin x+\cos x\right]=y\left[-x\sin x+\cos x\right]$ … (2)

$\therefore \quad y_2=y\left[-\left(x\cos x+\sin x\right)-\sin x\right]+\left(-x\sin x+\cos x\right)y_1$

$\Rightarrow \quad y_2=-y\left[x\cos x+2\sin x\right]+\left[-x\sin x+\cos x\right]y_1$ … (3)

And $\quad y_3=-y\left[-x\sin x+\cos x+2\cos x\right]-y_1\left[x\cos x+2\sin x\right]+\left[-x\sin x+\cos x\right]y_2$

$+y_1\left[-x\cos x-\sin x-\sin x\right]$

$=-y\left[-x\sin x+3\cos x\right]-y_1\left[x\cos x+2\sin x+x\cos x+2\sin x\right]+\left[-x\sin x+\cos x\right]y_2$

$y_3=-y\left[-x\sin x+3\cos x\right]-y_1\left[2x\cos x+4\sin x\right]+\left[-x\sin x+\cos x\right]y_2$ … (4)

Putting x = 0, in equation (1), (2), (3) and (4), we get

$y_0=e^{0\cos(0)}=1,$

$(y_1)_0=1[0+1]=1,$

$(y_2)_0=-1[0+0]+1[0+1]=1$

And $\left(y_3\right)_0 = -1[-0+3.1]-1[0+0]+[-0+1].1 = -2$

We know that by Maclaurin's theorem

$$f(x) = y_0 + \frac{x}{\lfloor 1}(y_1)_0 + \frac{x^2}{\lfloor 2}(y_2)_0 + \frac{x^3}{\lfloor 3}(y_3)_0 + \ldots.$$

$$\Rightarrow \quad e^{x\cos x} = 1 + x + \frac{x^2}{2}(1) + \frac{x^3}{6}(-2) + \ldots$$

$$\Rightarrow \quad e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots.$$

Thus, $\boxed{e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots.}$ **Answer**

b). Prove that the curvature at the point (*x*, y) of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ **is** $\frac{y^2}{c}$**.** **4**

Solution: Given the equation of curve is,

$$y = c\cosh\left(\frac{x}{c}\right) \qquad \ldots (1)$$

Differentiating *w.r.t.*, *x* we get

$$\frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right) \times \frac{1}{c} = \sinh\left(\frac{x}{c}\right)$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$$

Since, Radius of curvature in Cartesian form is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} \qquad \left[\because \cosh^2 x - \sinh^2 x = 1\right]$$

$$\Rightarrow \quad = c \times \frac{\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)} = c \times \frac{\cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}$$

$\Rightarrow \qquad = c \times \cosh^2\left(\frac{x}{c}\right) = c \times \left(\frac{y}{c}\right)^2 = \frac{y^2}{c}$ [From (1)]

Hence radius of curvature of curve is $\frac{y^2}{c}$. **Proved**

c). Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. **5**

Solution: Given: $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ … (1)

Partially differentiating both sides, *w.r.t.*, *x* and y , we get

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y \qquad \ldots (2)$$

and $$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y \qquad \ldots (3)$$

Taking, $\frac{\partial f}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0$

$$\Rightarrow x^3 - x + y = 0 \qquad \ldots (4)$$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0$

$$\Rightarrow y^3 + x - y = 0 \qquad \ldots (5)$$

Adding equation (4) and (5), we get

$$x^3 + y^3 = 0$$

$\Rightarrow \qquad (x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow \qquad x + y = 0$ but $x^2 - xy + y^2 \neq 0$

$\Rightarrow \qquad x = -y$ … (6)

Putting in equation (2), we get

$$x^3 - 2x = 0$$

$\Rightarrow \qquad x = 0,\ x = \sqrt{2}$

$\therefore \qquad y = 0,\ y = -\sqrt{2}$ [From (6)]

Thus the required stationary points are $(0, 0)$ and $(\sqrt{2}, -\sqrt{2})$

Again equation (2), partially differentiating *w.r.t.*, *x* and y, we get

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 \text{ and } s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

Equation (3), partially differentiating *w.r.t.*, y, we get

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

Case 1: at point $\left(\sqrt{2}, -\sqrt{2}\right)$

and $r = 12\left(\sqrt{2}\right)^2 - 4 = 20 > 0$, $s = 4$ and $t = 12\left(-\sqrt{2}\right)^2 - 4 = 20$

$\therefore \quad rt - s^2 = (20)(20) - (4)^2 = 384 > 0$

Therefore f(x, y) is minimum at $\left(\sqrt{2}, -\sqrt{2}\right)$.

Case 2: at Point (0, 0)

and $r = 12(0)^2 - 4 = -4 < 0$, $s = 4$ and $t = 12(0)^2 - 4 = -4$

$\therefore \quad rt - s^2 = (-4)(-4) - (4)^2 = 16 - 16 = 0$

The condition is doubt full and further investigation is needed.

3. a) If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$**, then prove that** $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$ **3**

Solution: Given $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$

$\Rightarrow \quad \sec u = \dfrac{x^3 - y^3}{x + y}$

Suppose z = sec$u$

$\therefore$ **t-test:** $z\,(xt, yt) = \dfrac{(t\,x)^3 - (t\,y)^3}{(t\,x) + (t\,y)} = t^2 \left(\dfrac{x^3 - y^3}{x + y} \right) = t^2\, z(x, y)$

Therefore, z be homogeneous function in x and y with degree 2, then by Euler theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow \quad x\frac{\partial}{\partial x}(\sec u) + y\frac{\partial}{\partial y}(\sec u) = 2(\sec u)$$

$$\Rightarrow \quad \sec u\,\tan u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right] = 2\sec u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\,\frac{1}{\cos u} \times \frac{1}{\sec u \tan u}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$$ **Answer**

b) The radius of a sphere is found to be 10cm with possible error of 0.02cm. what is the relative error in computing the volume. **4**

Solution: Suppose radius of sphere is r, then given

$r = 10cm$ and $\delta r = 0.02cm$

Since, Volume of sphere $V = \frac{4}{3}\pi r^3$

Taking log on both sides, we get

$$\log V = \log\left(\frac{4}{3}\right) + \log \pi + 3\log r$$

Differentiating on both sides, we get

$$\frac{\delta V}{V} = 0 + 0 + 3\left(\frac{\delta r}{r}\right)$$

$$\Rightarrow \quad E_r(V) = 3\left(\frac{\delta r}{r}\right) = 3\left(\frac{0.02}{10}\right) = 0.006$$

Thus, Relative error in volume of sphere is 0.006. **Answer**

c). If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$ **5**

Solution: Given $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$$

and $$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial\theta} = -r\sin\theta,\ \frac{\partial z}{\partial\phi} = 0$$

Now, $$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\theta\cos\phi\left[0 - r\sin\theta\cos\phi\cos\theta\right]$$
$$-r\sin\theta\sin\phi\left[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right]$$

$$= r^2\left[\sin^3\theta\cos^2\phi + \sin\theta\cos^2\phi\cos^2\theta + \sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta\sin^2\phi\right]$$

$$= r^2\left[\sin^3\theta\left(\cos^2\phi + \sin^2\phi\right) + \sin\theta\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$

$$= r^2\left[\sin^3\theta.1 + \sin\theta\left(1 - \sin^2\theta\right).1\right] = r^2\left[\sin^3\theta + \sin\theta - \sin^3\theta\right]$$

$$= r^2\sin\theta$$

Thus, $\boxed{\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta}$ **Hence proved**

4. a) Evaluate $\lim_{x\to\infty}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+\frac{9}{27+n^3}+\ldots+\frac{1}{2n}\right)$ **3**

Solution: Suppose

$$I=\lim_{x\to\infty}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+\frac{9}{27+n^3}+\ldots+\frac{1}{2n}\right)$$

$$\Rightarrow \quad =\lim_{x\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+\ldots+\frac{n^2}{(n+n)^3}\right]$$

$$\Rightarrow \quad =\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3+r^3} \qquad [r^{\text{th}}\text{ term}]$$

$\Rightarrow$ $$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} \frac{r^2}{1+\left(\frac{r}{n}\right)^3} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

Upper Limit: $\lim_{n\to\infty} \left[\frac{r}{n}\right]_{r=n} = \lim_{n\to\infty} \frac{n}{n} = 1$

Lower Limit: $\lim_{n\to\infty} \left[\frac{r}{n}\right]_{r=1} = 0$ (Fixed)

By Summation of series, we get

$$I = \int_0^1 \frac{x^2}{1+x^3}\, dx$$

Putting, $x^3 = t \Rightarrow x^2\, dx = dt/3$

$\Rightarrow$ $$= \frac{1}{3}\int_0^1 \frac{dt}{t+1} = \frac{1}{3}\left[\log(t+1)\right]_0^1$$

$\Rightarrow$ $$= \frac{1}{3}[\log 2 - 0] = \frac{1}{3}\log 2$$

Hence $\boxed{\lim_{x\to\infty}\left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} + \ldots + \frac{1}{2n}\right) = \frac{1}{3}\log 2}$ **Answer**

b) Prove that: $\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{a}, a > 0$ **4**

Solution: Given: $I = \int_{-\infty}^{\infty} e^{-a^2x^2} dx$

$\Rightarrow$ $$= 2\cdot\int_0^{\infty} e^{-a^2x^2} dx \qquad \ldots (1)$$ $\left[\because\ e^{-a^2x^2} \text{ is an even function}\right]$

Putting, $a^2x^2 = y$ so that $x = \frac{\sqrt{y}}{a}$

$\Rightarrow$ $$dx = \frac{dy}{2a\sqrt{y}}$$

From equation (1), we get

$$I = 2\int_0^\infty e^{-y}\,\frac{1}{2a\sqrt{y}}\,dy$$

$$\Rightarrow \quad = \frac{1}{a}\int_0^\infty e^{-y}\,y^{-\frac{1}{2}}$$

$$\Rightarrow \quad = \frac{1}{a}\int_0^\infty e^{-y}\,y^{\frac{1}{2}-1}\,dy = \frac{1}{a}\left\lceil\frac{1}{2}\right.$$

$$\Rightarrow \quad = \frac{\sqrt{\pi}}{a}$$

Thus, $\boxed{\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}}$ **Proved**

c). Express $\int_0^1 x^m\left(1-x^n\right)^p\,dx$ in terms of Beta functions and hence evaluate $\int_0^1 x^5\left(1-x^3\right)^{10}\,dx$ **5**

Solution: Given: $I = \int_0^1 x^m(1-x^n)^p\,dx$ … (1)

Putting, $x^n = t \quad \Rightarrow \quad x = t^{1/n}$

$$dx = \frac{t^{\left(\frac{1}{n}-1\right)}}{n}\,dt$$

$\therefore$ From equation (1), we get

$$I = \int_0^1 t^{\frac{m}{n}}(1-t)^p\,\frac{1}{n}\,t^{\left(\frac{1}{n}-1\right)}dt$$

$$\Rightarrow \quad = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^p\,dt$$

$$\Rightarrow \quad = \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{p+1-1}dt$$

$$\Rightarrow \quad I = \frac{1}{n}\beta\left(\frac{m+1}{n},\,p+1\right) \quad \ldots (2)$$

Putting, $m = 5$, $n = 3$ and $p = 10$, we get

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\,\beta\left(\frac{5+1}{3}, 10+1\right)$$

$$= \frac{1}{3}\beta(2,11) = \frac{1}{3}\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(2+11)}$$

$\Rightarrow$
$$= \frac{1}{3}\frac{1!\,10!}{12!} = \frac{1\,10!}{3\cdot 12\cdot 11\,10!} = \frac{1}{396}$$

Thus, $\boxed{\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{396}}$ **Answer**

5. a) Evaluate $\iint y\,dx$ over the part of the plane bounded by the line y = x and the parabola y = 4x – x^2.

Solution: Given: $I = \iint y\,dx$ … (1)

The equation of given curve are

y = x … (2)

and y = 4x – x^2 … (3)

From equation (2) and (3), we get

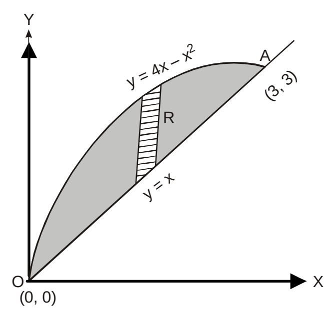
$$x = 4x - x^2 \quad \Rightarrow \quad x^2 - 3x = 0$$

$\Rightarrow$ $x = 0, x = 3$

$\therefore$ $y = 0, y = 3$ [From (2)]

Points of intersection of given curve are (0, 0) and (3, 3).

In the region R: x varies from 0 to 3 and y varies from x to $4x - x^2$.



$$\iint_R y\,dxdy = \int_0^3 \int_x^{4x-x^2} y\,dy\,dx$$

$\Rightarrow$
$$= \int_0^3 \left[\frac{y^2}{2}\right]_x^{4x-x^2} dx$$

$\Rightarrow$
$$= \frac{1}{2}\int_0^3 [(4x-x^2)^2 - (x)^2]\,dx$$

$\Rightarrow$
$$= \frac{1}{2}\int_0^3 (15x^2 + x^4 - 8x^3)\,dx$$

$\Rightarrow \qquad = \frac{1}{2}\left[5x^3 + \frac{x^5}{5} - 2x^4\right]_0^3 = \frac{1}{2}\left[405 + \frac{243}{5} - 162\right]$

$\Rightarrow \qquad = \frac{54}{5}$

Thus, $\boxed{\iint dx\,dy = \frac{54}{5}}$ **Answer**

b) Evaluate: $\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,y\,z\,dx\,dy\,dz$ **4**

Solution: : Suppose $I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$

$$= \int_0^1\int_0^{1-x} xy\left[\int_0^{1-x-y} z\,dz\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x} xy\frac{(1-x-y)^2}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{1-x} y[(1-x)^2 - 2(1-x)y + y^2]\,dy\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\int_0^{1-x} [(1-x)^2 y - 2(1-x)y^2 + y^3]\,dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x)^2\frac{y^2}{2} - 2(1-x)\frac{y^3}{3} + \frac{y^4}{4}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^4}{2} - 2\frac{(1-x)^4}{3} + \frac{(1-x)^4}{4}\right]dx$$

$$= \frac{1}{24}\int_0^1 x(1-x)^4\,dx$$

Putting, $1-x = t \;\; dx = -dt$

$$= \frac{1}{24}\int_1^0 (1-t)t^4(-dt) = \frac{1}{24}\int_0^1 (t^4 - t^5)\,dt$$

$$= \frac{1}{24}\left[\frac{t^5}{5} - \frac{t^6}{6}\right]_0^1 = \frac{1}{24}\left[\frac{1}{5} - \frac{1}{6}\right] = \frac{1}{24}\frac{[6-5]}{30}$$

$$= \frac{1}{720}$$

Thus $\boxed{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dx\, dy\, dz = \frac{1}{720}}$

c). Find the area enclosed by the parabolas y^2 = 4*ax* and *x*^{2} = 4*a*y. **5**

Solution: Given: the equations of parabola are

$$y^2 = 4ax \qquad \ldots(1)$$

and $$x^2 = 4ay \qquad \ldots(2)$$

Squaring both sides, we get

$$x^4 = 16a^2y^2$$

$\Rightarrow$ $$x^4 = 16a^2(4ax)$$

$\Rightarrow$ $$x\left(x^3 - 64a^3\right) = 0$$

$\Rightarrow$ $$x = 0 \text{ and } x^3 = 64a^3$$

$\Rightarrow$ $$x = 0 \text{ and } x = 4a$$

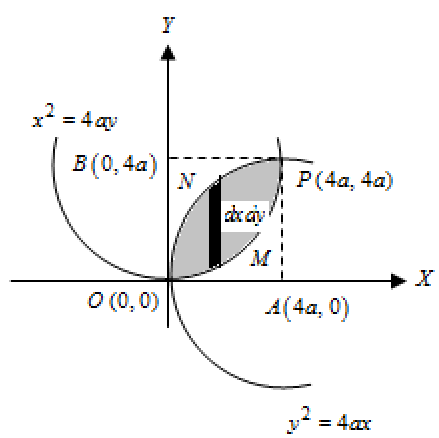
Putting in equation (1), we get

$$y = 0 \text{ and } y = 4a$$

Therefore required point of intersection are (0, 0) and (4a, 4a).

Now, we will draw the rough sketch for the Region R (OMPNO).

In the region R, the limits are



Limits of y : $y = \frac{x^2}{4a}$ to $2\sqrt{ax}$

Limit of x : $x = 0$ to $x = 4a$

$\therefore$ Area between the parabolas A = Area of Region R

$\Rightarrow$ $$= \iint_R 1.\, dx\, dy$$

$\Rightarrow \quad = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dx\, dy$

$\Rightarrow \quad = \int_{x=0}^{4a} \left[y \right]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx = \int_{x=0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$

$\Rightarrow \quad = 2\sqrt{a} \left[\frac{2}{3} x^{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$

$\Rightarrow \quad = \frac{4\sqrt{a}}{3} \left[(4a)^{3/2} - 0 \right] - \frac{1}{12a} \left[(4a)^3 - 0 \right]$

$\Rightarrow \quad = \frac{32}{3} a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}$ **Answer**

6. a) Evaluate $\int_a^b e^x \, dx$ as limit of sum. **3**

Solution: Suppose $f(x) = e^x$ and $nh = b - a$

We know that by definition of definite integral as limit of sum,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h \left[f(a) + f(a+h) + f(a+2h) + \ldots. + f(a+(n-1)h) \right]$$

$\Rightarrow \quad \int_a^b e^x \, dx = \lim_{h \to 0} h \left[e^a + e^{(a+h)} + e^{(a+2h)} + \ldots. + e^{(a+n-1h)} \right]$

$\Rightarrow \quad = e^a \lim_{h \to 0} h \left[1 + e^h + e^{2h} + \ldots + e^{(n-1)h} \right]$

$\Rightarrow \quad = e^a \lim_{h \to 0} h \left[\frac{1\left(e^{nh} - 1\right)}{e^h - 1} \right] \qquad \left[\because \ S_n = \frac{a\left(r^n - 1\right)}{r - 1}, r > 1 \right]$

$\Rightarrow \quad = e^a \lim_{h \to 0} h \left[\frac{1\left(e^{b-a} - 1\right)}{e^h - 1} \right] = e^a \left(e^{b-a} - 1 \right) \left(\lim_{h \to 0} \frac{h}{e^h - 1} \right)$

$\Rightarrow \quad = \left(e^b - e^a \right) \times 1 = e^b - e^a$

Thus $\boxed{\int_a^b e^x\, dx = e^b - e^a}$ **Answer**

b) Express in terms of the Gamma function: $\int_0^\infty x^n\, e^{-a^2x^2}\, dx$ **4**

Solution: Given: $I = \int_0^\infty x^n\, e^{-a^2x^2}\, dx$ … (1)

Putting, $a^2x^2 = t \Rightarrow x = \frac{\sqrt{t}}{a}$

$$\Rightarrow dx = \frac{dt}{2a\sqrt{t}}$$

Upper Limit: $t = a^2(\infty)^2 = \infty$

Lower Limit: $t = a^2(0)^2 = 0$

$\therefore$ From (1), we get

$$I = \int_0^\infty \left(\frac{\sqrt{t}}{a}\right)^n e^{-t}\, \frac{dt}{2a\sqrt{t}}$$

$$\Rightarrow \quad = \frac{1}{2a^{n+1}} \int_0^\infty e^{-t}\, t^{\frac{n}{2}-\frac{1}{2}}\, dt$$

$$\Rightarrow \quad = \frac{1}{2a^{n+1}} \int_0^\infty e^{-t}\, t^{\left(\frac{n}{2}-\frac{1}{2}+1\right)-1}\, dt$$

$$\Rightarrow \quad = \frac{1}{2a^{n+1}} \int_0^\infty e^{-t}\, t^{\left(\frac{n+1}{2}\right)-1}\, dt$$

$$\Rightarrow \quad = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$$

Thus, $\boxed{\int_0^\infty x^n\, e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)}$ **Proved**

c) Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} x\,y\,dx\,dy$ **5**

Solution: $I = \int_0^1 \int_{x^2}^{2-x} x\,y\,dx\,dy$ … (1)

The limits of integration are

(i). $x = 0$, i.e., Y-axis

(ii). $x = 1$, i.e., the equation straight line parallel to Y-axis

(iii). $y = x^2$ i.e., Equation of parabola symmetric about Y-axis and vertex at (0, 0).

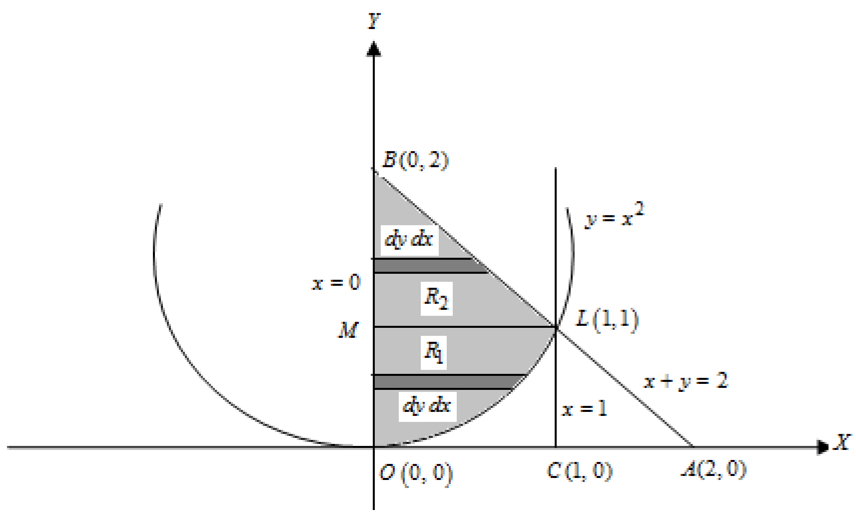
(iv). $y = 2 - x$ i.e. $\frac{x}{2}+\frac{y}{2}=1$, equation of straight line, which form the equal intercept on both the axes and intersect X-axis at (2, 0) and Y-axis at (0, 2).

Point of intersection of straight line $x + y = 2$, parabola $y = x^2$ and $x = 1$:

Putting, x = 1 in $y = x$ and parabola $y = x^2$, then we get y = 1

∴ Point of intersection is (1, 1)

Now, we will draw the rough sketch curve for region R.



In above figure, we divided the bounded region R, into subdivided region R_1 and R_2 and draw the strip parallel to X-axis say *dydx*.

Case 1: In the region R_1 (OLMO)

Limit of x : $x = 0$ to $x = \sqrt{y}$

Limit of y : $y = 0$ to $y = 1$ [Point 0 (0, 0) to point L (1, 1)]

∴ The change of order of integration in the region R_1 is

$$I_1 = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} x\,y\,dy\,dx \qquad \ldots (2)$$

Case 2: In the region R_2 (MLBM)

Limit of x : $x=0$ to $x=2-y$

Limit of y : $y=1$ to $y=2$ [Point L (1, 1) to point B (0, 2)]

$\therefore$ The change of order of integration in the region R_2 is

$$I_2 = \int_{y=1}^{2}\int_{x=0}^{2-y} x\,y\,dy\,dx \qquad \ldots (3)$$

Therefore, from equation (2) and (3), we get

$$I = I_1 + I_2$$

$$\int_0^1\int_{x^2}^{2-x} x\,y\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} x\,y\,dy\,dx + \int_{y=1}^{2}\int_{x=0}^{2-y} x\,y\,dy\,dx$$

This is required change of order of given integral (1).

Next, we will evaluate it.

$$= \int_0^1\left[\int_0^{\sqrt{y}} xy\,dx\right]dy + \int_1^2\left[\int_0^{2-y} xy\,dx\right]dy$$

$$\Rightarrow \quad = \int_0^1\left[\frac{x^2y}{2}\right]_0^{\sqrt{y}} dy + \int_1^2\left[\frac{x^2y}{2}\right]_0^{2-y} dy$$

$$\Rightarrow \quad = \int_0^1\left(\frac{y^2}{2}\right)dy + \frac{1}{2}\int_1^2 y\cdot(2-y)^2\,dy$$

$$\Rightarrow \quad = \frac{1}{6}[y^3]_0^1 + \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)\,dy$$

$$\Rightarrow \quad = \frac{1}{6} + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4}{3}y^3\right]_1^2$$

$$\Rightarrow \quad = \frac{1}{6} + \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right]$$

$$\Rightarrow \quad = \frac{1}{6} + \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right]$$

$\Rightarrow \qquad = \frac{1}{6} + \frac{1}{24}[72 + 45 - 112]$

$\Rightarrow \qquad = \frac{1}{6} + \frac{5}{24}$

$\Rightarrow \qquad = \frac{3}{8}$

Thus, $\boxed{\int_0^1 \int_{x^2}^{2-x} x\,y\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} x\,y\,dy\,dx + \int_{y=1}^{2}\int_{x=0}^{2-y} x\,y\,dy\,dx = \frac{3}{8}}$ **Answer**

7. a) Verify Rolle's theorem, where $f(x) = 2x^3 + x^2 - 4x - 2$. **3**

Solution: Given the function is

$$f(x) = 2x^3 + x^2 - 4x - 2 \qquad \ldots (1)$$

Taking, $f(x) = 0$

$\Rightarrow \qquad 2x^3 + x^2 - 4x - 2 = 0$

$\Rightarrow \quad x^2(2x+1)-2(x+2)=0$ i.e., $(2x+1)(x^2-2)=0$

$\Rightarrow \quad x=-1, x=-\sqrt{2}, x=\sqrt{2}$

$\therefore$ The required interval is $\left[-\sqrt{2}, \sqrt{2}\right]$

(i). Putting $x=-\sqrt{2} \Rightarrow f\left(-\sqrt{2}\right)=0$ and $x=\sqrt{2} \Rightarrow f\left(\sqrt{2}\right)=0$

Clearly $f\left(-\sqrt{2}\right)=f\left(\sqrt{2}\right)=0$

(ii). Since $f(x)$ is a polynomial function in x, then $f(x)$ is continuous in $\left[-\sqrt{2}, \sqrt{2}\right]$

(iii). Since $f(x)$ is a polynomial function in x, then it can differentiate such that

$f'(x)=6x^2+2x-4$

then by Rolle's theorem $\exists$ at least $c \in \left(-\sqrt{2}, \sqrt{2}\right)$ such that

$$f'(c)=0$$

$\Rightarrow \quad 6c^2+2c-4=0$ i.e. $3c^2+c-2=0$

$\Rightarrow \quad 3c^2+3c-2c-2=0$ i.e. $3c(c+1)-2(c+1)=0$

$\Rightarrow \quad (3c-2)(c+1)=0$ i.e., $c=\dfrac{2}{3}, -1 \in \left(-\sqrt{2}, \sqrt{2}\right)$

Hence verified Rolle's theorem for $\left[-\sqrt{2}, \sqrt{2}\right]$.

b) If $u=f(y-z, z-x, x-y)$**, prove that:** $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$ **4**

Solution: Suppose $X=y-z,\ Y=z-x, Z=x-y$, then $u=f(X, Y, Z)$

Therefore u is composite function x, y and z respectively.
We have $X=y-z,\ Y=z-x, Z=x-y$
Partially differentiate *w.r.t.* x, y and z respectively, we get

$$\frac{\partial X}{\partial x}=0, \frac{\partial X}{\partial y}=1, \frac{\partial X}{\partial z}=-1$$

And $$\frac{\partial Y}{\partial x}=-1, \frac{\partial Y}{\partial y}=0, \frac{\partial Y}{\partial z}=1, \frac{\partial Z}{\partial x}=1, \frac{\partial Z}{\partial y}=-1, \frac{\partial Z}{\partial z}=0$$

Now, $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}.\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X}.(0) + \frac{\partial u}{\partial Y}.(-1) + \frac{\partial u}{\partial Z}.(1) = -\frac{\partial u}{\partial Y}.+\frac{\partial u}{\partial Z} \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X}.(1) + \frac{\partial u}{\partial Y}.(0) + \frac{\partial u}{\partial Z}.(-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}.(-1) + \frac{\partial u}{\partial Y}.(1) + \frac{\partial u}{\partial Z}.(0) = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad \ldots (3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y}.+\frac{\partial u}{\partial Z} + \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} = 0$$

$\Rightarrow$ $$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$ **Hence proved**

c). Trace the curve $y^2(2a-x)=x^3$ **5**

Solution: Given the equation of curve is

$$y^2(2a-x)=x^3 \qquad \ldots (1)$$

1. Symmetry:

In the curve power of y is an even, therefore curve symmetric about X-axis.

2. Nature of origin:

Equation of curve satisfies through the origin (0, 0), therefore curve passes through the origin.

3. Tangent at origin:

Equating the lowest power term to zero, *i.e.* $y^2 = 0 \Rightarrow y = 0, 0$.

Therefore origin is a double point and it is a cusp.

4. Point of intersection with coordinate axes:

When $x = 0$, then y = 0 and when y = 0, then $x = 0$. Thus the curve meets the coordinate axes only at origin.

5. Region of Existence and Nature of curve:

From (1), we have $y^2 = \dfrac{x^3}{2a-x}$

(I). When $x > 0$

(i). If $0 < x < 2a$, then y^2 = + ve and y is real,

(ii). If $x > 2a$, then y^2 = –ve $\Rightarrow$ y is imaginary.

(II). When $x < 0$, then y^2 = –ve $\Rightarrow$ y is imaginary.

Thus the curve lies only in the region $0 < x < 2a$

6. Asymptotes:

Equation (1), can be written in the form,

$$x\left(x^2+y^2\right)=2ay^2 \quad \dots (2)$$

If asymptotes parallel to Y-axis, then equating the coefficient of highest power of y equal to zero i.e.

$x-2a=0 \Rightarrow x=2a$

7. Special point:

Since X-axis is common tangent to the two branches of curve passing through origin hence origin is a cusp.

8. Rough sketch of curve:

From the above points, the shape of the curve is as shown in the figure